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KINEMATICS OF FINITE-STRAIN ELASTIC–INELASTIC DEFORMATION

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Polar decomposition tensors are constructed for slightly disturbed kinematic elastic, inelastic, and thermal strain tensors. Provided that the inelastic and thermal site gradients are pure deformations without rotations, relations are obtained between inelastic small strains and small rotations and between thermal small strains and small rotations which transform an intermediate configuration to a close current configuration.

Key words: *elastic, inelastic, and thermal strains, small disturbances, eigenvectors and eigenvalues, polar decomposition.*

Introduction. In [1–4], the kinematics of elastic–inelastic and thermo-elastic–inelastic deformation processes is described by the relation

$$F = f \cdot F_*. \tag{1}$$

Here F, f, and F_* are the elastic–inelastic (thermo-elastic–inelastic) site gradients which transform the initial configuration to the current one, an intermediate configuration close to the current one to the actual configuration, and the initial configuration to the intermediate one. In turn, $f = f_{\rm E} \cdot f_{\rm IN} \cdot f_\Theta$, where $f_{\rm E}$, $f_{\rm IN}$, and f_Θ are the elastic, inelastic, and thermal site gradients, each of which is defined by the relation $f_i = g + \varepsilon h_i$ (the subscript i refer to E, IN, or Θ), g is a unit tensor, ε is a small positive quantity that characterizes the closeness of the intermediate and current configurations, and h_i is the gradient of the elastic, inelastic, and thermal vectors of small displacements $\boldsymbol{u}_i$ relative to the intermediate configuration. This representation admits commutation of the site gradients $f_{\rm E}$, $f_{\rm IN}$, and f_Θ allows f to be represented as $f = g + \varepsilon h$, where $h = h_{\rm E} + h_{\rm IN} + h_\Theta$ is the total-displacement gradient (accurate up to terms linear in ε). Each displacement gradient is represented in terms of the symmetric part e, $e_{\rm E}$, $e_{\rm IN}$, and e_Θ (small deformations) and the skew-symmetric part d, $d_{\rm E}$, $d_{\rm IN}$, and d_Θ (small rotations), $e = e_{\rm E} + e_{\rm IN} + e_\Theta$, and $d = d_{\rm E} + d_{\rm IN} + d_\Theta$ (these and only these small total deformations and rotations are compatible).

Relation (1) reflects the history of the process, i.e., any sequence and duration of purely elastic, purely inelastic, and purely thermal deformations. The gradient F_* is normalized to the time t_*, the gradient F to the current time t, and $t - t_* = \varepsilon\tau$ ($\tau > 0$). In [3, 4], the kinematics (1) was split into the thermal kinematic F_Θ, inelastic kinematic $F_{\rm IN}$, and purely elastic kinematic $F_{\rm E}$ using the notions of matricant and multiplicative integral. As a result, relation (1) is represented as

$$F = F_{\rm E} \cdot F_{\rm IN} \cdot F_\Theta = [g + \varepsilon(h_{\rm E} + h_{\rm IN} + h_\Theta)] \cdot F_*, \qquad F_* = F_{{\rm E}*} \cdot F_{{\rm IN}*} \cdot F_{\Theta*}, \tag{2}$$

where the site gradients F, $F_{\rm E}$, $F_{\rm IN}$, and F_Θ are determined at the current time t, and the same gradients denoted by asterisk $*$ are determine at the time t_*. Representation (2) is similar in shape to the well-known Lie expansion but is free from the drawbacks of the latter [3, 4]. The expressions for $F_{\rm E}$, $F_{\rm IN}$, and F_Θ obtained in [3, 4] are written as

$$F_{\rm E} = (g + \varepsilon h_{\rm E}) \cdot F_{{\rm E}*}, \qquad F_{\rm IN} = (g + \varepsilon F_{{\rm E}*}^{-1} \cdot h_{\rm IN} \cdot F_{{\rm E}*}) \cdot F_{{\rm IN}*},$$
$$F_\Theta = (g + \varepsilon F_{{\rm IN}*}^{-1} \cdot F_{{\rm E}*}^{-1} \cdot h_\Theta \cdot F_{{\rm E}*} \cdot F_{{\rm IN}*}) \cdot F_{\Theta*}. \tag{3}$$

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We note that in (2) and (3), the subscripts IN and Θ can be interchanged. It is convenient to write relations (3) as

$$F_i = (g + \varepsilon P_i) \cdot F_{i*}, \qquad P_i = \begin{cases} h_{\mathrm{E}}, & i \equiv E, \\ F_{\mathrm{E}*}^{-1} \cdot h_{\mathrm{IN}} \cdot F_{\mathrm{E}*}, & i \equiv IN, \\ F_{\mathrm{IN}*}^{-1} \cdot F_{\mathrm{E}*}^{-1} \cdot h_\Theta \cdot F_{\mathrm{E}*} \cdot F_{\mathrm{IN}*}, & i \equiv \Theta. \end{cases} \tag{4}$$

All relations and equations of continuum mechanics should satisfy the principle of objectivity, i.e., they should be materially independent of the frame of reference in which the motion is considered. It has been shown [4] that, for expansion (2), the principle of objectivity is satisfied for all relations if the site gradients F_{IN} and F_Θ are pure deformations without rotations: $F_{\mathrm{IN}} = U_{\mathrm{IN}} = V_{\mathrm{IN}}$ and $F_\Theta = U_\Theta = V_\Theta$, i.e., if in the polar decomposition of the tensors $F_{\mathrm{IN}} = R_{\mathrm{IN}} \cdot U_{\mathrm{IN}} = V_{\mathrm{IN}} \cdot R_{\mathrm{IN}}$ and $F_\Theta = R_\Theta \cdot U_\Theta = V_\Theta \cdot R_\Theta$, the orthogonal tensors R_{IN} and R_Θ are unity tensors: $R_{\mathrm{IN}} = R_\Theta = g$. The latter condition should define the missing relations between e_{IN} and d_{IN} and between e_Θ and d_Θ in Eqs. (3) and (4). These relations are missing because constitutive equations are known only for small inelastic and thermal strains (rates). This, for example, is the associated law in the case of plasticity [1], the differential law $\dot{e}_{\mathrm{IN}} = T/\mu$ (T is the stress tensor and μ is the viscosity) in the case of viscosity [2], and the law $\dot{e}_\Theta = \beta\dot{\Theta} g$ (β is the linear thermal-expansion coefficient and Θ is the absolute temperature) in the case of thermo-elasticity. For small rotations (rates), relations of this type are absent. The present paper, which is a continuation of [4], seeks to establish such relations. In addition, it is of interest to determine the structure of the elastic, inelastic, and thermal kinematic tensors of pure strain and pure rotation and the laws of their variation with variation of their corresponding site gradients.

Taking into account that the relationship between the kinematic tensors F, R, U, and V corresponding to the total, elastic, inelastic or thermal strains is given by the relations

$$U = \sum_{i=1}^{3} U_i \boldsymbol{\delta}_i^{(1)} \boldsymbol{\delta}_i^{(1)}, \qquad V = \sum_{i=1}^{3} U_i \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_i^{(2)}, \qquad R = \sum_{i=1}^{3} \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_i^{(1)}, \qquad F = \sum_{i=1}^{3} U_i \, \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_i^{(1)},$$

where U_i are the eigenvalues of the symmetric positive definite tensor U (or V), $\boldsymbol{\delta}_i^{(1)}$ are the unit and orthogonal eigenvectors of the tensor U, and $\boldsymbol{\delta}_i^{(2)}$ are the unit and orthogonal eigenvectors of the tensor V, we consider their variation under small disturbances, i.e., for transition from one configuration (intermediate) to another, fairly close (current) configuration.

An intermediate configuration close to the current configuration is introduced to linearize the equations in solving nonlinear boundary-value problems, in particular, problems of the nonlinear theory of elasticity (see, for example, [5]). This approach has also proved effective in constructing kinematic and constitutive equations for elastic–inelastic media since it allows one to obtain nonholonomic (unrepresentable in finite integral form, unlike for the case of nonlinear elasticity) evolutionary relations which reflect the history of the process [1–4]. While in the case of nonlinear elasticity, the introduction of an intermediate configuration close the current one is a convenient tool for the numerical implementation of boundary-value problems, in the case of elastic–inelastic environments, this approach provides, in addition, an effective apparatus for constructing kinematic and constitutive relations. This apparatus is used in this work to study the variation of the indicated kinematic tensors.

1. Eigenvalues and Eigenvectors of a Slightly Disturbed Symmetric Positive Definite Tensor of the Second Rank. Let A be a symmetric positive definite tensor of the second rank which is represented in terms of eigenvalues and eigenvectors as

$$A = \sum_{i=1}^{3} A_i \boldsymbol{\delta}_i \boldsymbol{\delta}_i, \tag{1.1}$$

where $A_i > 0$ and $\boldsymbol{\delta}_i$ are unit orthogonal vectors. We impart a small disturbance εa (ε is a small positive quantity) to the tensors A such that the tensor $A' = A + \varepsilon a$ is also symmetric and positive definite (from this, it follows that the tensor a is at least symmetric). Then, by virtue of the small disturbances and continuity of the tensor A, the eigenvalues and eigenvectors of the tensor A' can be represented as $A'_i = A_i + \varepsilon\lambda_i$ and $\boldsymbol{\delta}'_i = (g + \varepsilon d_a) \cdot \boldsymbol{\delta}_i$. Here $A'_i > 0$, $\boldsymbol{\delta}'_i$ are unit orthogonal vectors, $\varepsilon\lambda_i$ and $\varepsilon d_a \cdot \boldsymbol{\delta}_i$ are small changes in the eigenvalues and eigenvectors of the tensor A, and d_a is a skew-symmetric tensor. It is only in this case that the tensor $g + \varepsilon d_a$ is orthogonal and rotates the three orthonormal vectors $\boldsymbol{\delta}_i$ in space without changing their length and the angles between them. The orthogonality of the tensor $g + \varepsilon d_a$ follows from the following. On the one hand, $(g + \varepsilon d_a)^{\mathrm{t}} = g - \varepsilon d_a$ since

$d_a^{\mathrm{t}} = -d_a$; on the other hand, from the identity $(g+\varepsilon d_a)^{-1}\cdot(g+\varepsilon d_a) = g$, representing the tensor $(g+\varepsilon d_a)^{-1}$ as $m+\varepsilon n$, we obtain the equality $m+\varepsilon(n+m\cdot d_a) = g$ accurate up to linear terms in ε. Equating the coefficients at the same powers of ε, we have $m = g$ and $n = -d_a$. From this, it follows that $(g+\varepsilon d_a)^{\mathrm{t}} = (g+\varepsilon d_a)^{-1}$, i.e., the tensor $g+\varepsilon d_a$ is orthogonal.

In view of the aforesaid, the eigenvalue problem for the tensor A' reduces to three vector relations

$$[A - A_i g + \varepsilon(A - A_i g)\cdot d_a + \varepsilon(a - \lambda_i g)]\cdot\boldsymbol{\delta}_i = 0, \qquad i = 1,2,3.$$

Taking into account that $(A - A_i g)\cdot\boldsymbol{\delta}_i = 0$ (at $i = 1,2,3$) and representing the tensor A in the form of (1.1), and the unit tensor in the basis $\boldsymbol{\delta}_i$, we reduce these three vector relations to nine scalar vectors:

$$(A_j - A_i)(\boldsymbol{\delta}_j\cdot d_a\cdot\boldsymbol{\delta}_i) + [(\boldsymbol{\delta}_j\cdot a\cdot\boldsymbol{\delta}_i) - \lambda_i\delta_{ij}] = 0, \qquad i,j = 1,2,3$$

(δ_{ij} is Kronecker symbol), three of which are equivalent. As a result, the system is split into six equations:

$$(A_i - A_j)(\boldsymbol{\delta}_i\cdot d_a\cdot\boldsymbol{\delta}_j) + \boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_j = 0, \qquad i = 1,2,3, \quad j = \begin{cases} i+1, & i<3, \\ 1, & i=3, \end{cases}$$
$$\lambda_i = \boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_i, \qquad i = 1,2,3, \tag{1.2}$$

from which, in the case of different eigenvalues A_i, we find λ_i and the components of the skew-symmetric tensor d_a in the basis $\boldsymbol{\delta}_i$:

$$d_a = d_a^{12}(\boldsymbol{\delta}_1\boldsymbol{\delta}_2 - \boldsymbol{\delta}_2\boldsymbol{\delta}_1) + d_a^{23}(\boldsymbol{\delta}_2\boldsymbol{\delta}_3 - \boldsymbol{\delta}_3\boldsymbol{\delta}_2) + d_a^{31}(\boldsymbol{\delta}_3\boldsymbol{\delta}_1 - \boldsymbol{\delta}_1\boldsymbol{\delta}_3), \qquad d_a^{ij} = -\frac{\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_j}{A_i - A_j}. \tag{1.3}$$

If two eigenvalues are equal, for example, $A_1 = A_2 \neq A_3$, only the unit vector $\boldsymbol{\delta}_3$ is uniquely determined. The unit vectors $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are orthogonal to each other and to the vector $\boldsymbol{\delta}_3$ and are unique up to rotation around $\boldsymbol{\delta}_3$. The condition $\boldsymbol{\delta}_1\cdot a\cdot\boldsymbol{\delta}_2 = 0$, which follows from (1.2), relates this rotation to the symmetric tensor a; in this case, relation (1.3) implies the arbitrariness of the component d_a^{12} of the skew-symmetric tensor d_a. If all eigenvalues are equal, three unit orthogonal vectors $\boldsymbol{\delta}_i$ are related to the tensor a by the conditions

$$\boldsymbol{\delta}_1\cdot a\cdot\boldsymbol{\delta}_2 = 0, \qquad \boldsymbol{\delta}_2\cdot a\cdot\boldsymbol{\delta}_3 = 0, \qquad \boldsymbol{\delta}_3\cdot a\cdot\boldsymbol{\delta}_1 = 0,$$

and all components of the skew-symmetric tensor d_a are arbitrary.

The tensor A' can be represented as

$$A' = A + \varepsilon a = \sum_{i=1}^{3}[A_i + \varepsilon(\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_i)][(g+\varepsilon\, d_a)\cdot\boldsymbol{\delta}_i\boldsymbol{\delta}_i\cdot(g - \varepsilon d_a)]. \tag{1.4}$$

From this expression, keeping only terms linear in ε, we have

$$A' = A + \varepsilon a = A + \varepsilon d_a\cdot A - \varepsilon A\cdot d_a + \varepsilon\sum_{i=1}^{3}(\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_i)\boldsymbol{\delta}_i\boldsymbol{\delta}_i. \tag{1.5}$$

Substitution of the expression for d_a into (1.5) yields the relation

$$A + \varepsilon a = A + \varepsilon(\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_j)\boldsymbol{\delta}_i\boldsymbol{\delta}_j,$$

i.e., identity.

Because the tensor A' is positive definite, there exists a tensor $(A')^{1/2}$, which is represented in the form of (1.4) with eigenvalues to power 1/2:

$$[A_i + \varepsilon(\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_i)]^{1/2} = A_i^{1/2} + (\varepsilon/2)A_i^{-1/2}(\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_i)$$

(in the series expansion, only terms linear in ε are kept). As a result, the linearized expression for the tensor $(A')^{1/2}$ is written in form similar to (1.5):

$$(A')^{1/2} = (A+\varepsilon a)^{1/2} = A^{1/2} + \varepsilon d_a\cdot A^{1/2} - \varepsilon A^{1/2}\cdot d_a + \frac{\varepsilon}{2}\sum_{i=1}^{3}A_i^{-1/2}(\boldsymbol{\delta}_i\cdot a\cdot\boldsymbol{\delta}_i)\boldsymbol{\delta}_i\boldsymbol{\delta}_i.$$

Substitution of the expression for d_a into the above expression yields the elegant relation

$$(A')^{1/2} = (A + \varepsilon a)^{1/2} = A^{1/2} + \varepsilon \, \frac{\boldsymbol{\delta}_i \cdot a \cdot \boldsymbol{\delta}_j}{A_i^{1/2} + A_j^{1/2}} \, \boldsymbol{\delta}_i \boldsymbol{\delta}_j. \tag{1.6}$$

The approximate relation (1.6) (in which only terms linear in ε are kept) can be reduced to the exact evolutionary relation. Rearranging $A^{1/2}$ to the left side, dividing the entire expression by ε, and letting ε tend to zero, we obtain the Gâteaux derivative

$$D(A^{1/2}) = \frac{\boldsymbol{\delta}_i \cdot \dot{a} \cdot \boldsymbol{\delta}_j}{A_i^{1/2} + A_j^{1/2}} \, \boldsymbol{\delta}_i \boldsymbol{\delta}_j. \tag{1.7}$$

Here all quantities are determined at the time t.

2. Small Disturbances of Kinematic Elastic, Inelastic, and Thermal Strain Tensors and Their Corresponding Polar Decomposition Tensor. We concretize the above relations with respect to the kinematic tensors of the Cauchy–Green (C) and Finger (Φ) strain measures. For the tensor $F_{\rm E}$ (3), these measures are represented as

$$C_{\rm E} = F_{\rm E}^{\rm t} \cdot F_{\rm E} = C_{{\rm E}*} + 2\varepsilon F_{{\rm E}*}^{\rm t} \cdot e_{\rm E} \cdot F_{{\rm E}*}, \qquad C_{{\rm E}*} = F_{{\rm E}*}^{\rm t} \cdot F_{{\rm E}*},$$

$$\Phi_{\rm E} = F_{\rm E} \cdot F_{\rm E}^{\rm t} = \Phi_{{\rm E}*} + \varepsilon (h_{\rm E} \cdot \Phi_{{\rm E}*} + \Phi_{{\rm E}*} \cdot h_{\rm E}^{\rm t}), \qquad \Phi_{{\rm E}*} = F_{{\rm E}*} \cdot F_{{\rm E}*}^{\rm t}$$

or, using the tensor $F_{\rm E}$ of the polar representation ($F_{\rm E} = R_{\rm E} \cdot U_{\rm E} = V_{\rm E} \cdot R_{\rm E}$, where $R_{\rm E}$ is an orthogonal tensor and $U_{\rm E}$ and $V_{\rm E}$ are the right and left symmetric positive definite pure strain tensors), they are represented as

$$U_{\rm E}^2 = U_{{\rm E}*}^2 + 2\varepsilon F_{{\rm E}*}^{\rm t} \cdot e_{\rm E} \cdot F_{{\rm E}*}, \qquad V_{\rm E}^2 = V_{{\rm E}*}^2 + \varepsilon (h_{\rm E} \cdot V_{{\rm E}*}^2 + V_{{\rm E}*}^2 \cdot h_{\rm E}^{\rm t}). \tag{2.1}$$

We associate these tensors with the tensors A', A, and a from expression (1.4). If elastic kinematics is considered and the tensors are represented using the intermediate configuration, the subscripts E and $*$ are omitted for simplicity. Then, in the first relation (2.1), the tensor $U_{\rm E}^2$ [we denote it by $(U')^2$] corresponds to the tensor A' in expression (1.4), the tensor U^2 ($A = U^2$) with the eigenvalues $A_i = U_i^2$ and the eigenvectors $\boldsymbol{\delta}_i = \boldsymbol{\delta}_i^{(1)}$ $\Big(U = \sum\limits_{i=1}^{3} U_i \boldsymbol{\delta}_i^{(1)} \boldsymbol{\delta}_i^{(1)}\Big)$ corresponds to the tensor A, and the tensor $2F^{\rm t} \cdot e \cdot F$ to the tensor a. In the second relation (2.1), the tensor $V_{\rm E}^2$ [we denote it by $(V')^2$] corresponds to the tensor A', the tensor V^2 ($A = V^2$) played by the with the same eigenvalues $A_i = U_i^2$ as in the first relation but with the eigenvectors $\boldsymbol{\delta}_i = \boldsymbol{\delta}_i^{(2)}$ $\Big(V = \sum\limits_{i=1}^{3} U_i \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_i^{(2)}\Big)$ corresponds to the tensor A, and the tensor $h \cdot V^2 + V^2 \cdot h^{\rm t}$ to the tensor a. As is known, the orthogonal tensor R transforms the vectors $\boldsymbol{\delta}_i^{(1)}$ to the vectors $\boldsymbol{\delta}_i^{(2)}$ $\Big(\boldsymbol{\delta}_i^{(2)} = R \cdot \boldsymbol{\delta}_i^{(1)} \Rightarrow R = \sum\limits_{i=1}^{3} \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_i^{(1)}\Big)$ and the tensor F is represented as $F = \sum\limits_{i=1}^{3} U_i \, \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_i^{(1)}$. In view of the aforesaid, expression (1.6) for the tensors U' and V' is written as

$$U' = U + 2\varepsilon \, \frac{U_i (\boldsymbol{\delta}_i^{(2)} \cdot e \cdot \boldsymbol{\delta}_j^{(2)}) U_j}{U_i + U_j} \, \boldsymbol{\delta}_i^{(1)} \boldsymbol{\delta}_j^{(1)}, \qquad V' = V + \varepsilon \, \frac{U_i^2 (\boldsymbol{\delta}_i^{(2)} \cdot h^{\rm t} \cdot \boldsymbol{\delta}_j^{(2)}) + (\boldsymbol{\delta}_i^{(2)} \cdot h \cdot \boldsymbol{\delta}_j^{(2)}) U_j^2}{U_i + U_j}, \tag{2.2}$$

and expression (1.3), which defines the transformation of the vectors $\boldsymbol{\delta}_i^{(1)}$ to $\boldsymbol{\delta}_i^{(1)'}$ and $\boldsymbol{\delta}_i^{(2)}$ to $\boldsymbol{\delta}_i^{(2)'}$ keeping them unit and orthogonal, is written as

$$d_U = d_U^{12} (\boldsymbol{\delta}_1^{(1)} \boldsymbol{\delta}_2^{(1)} - \boldsymbol{\delta}_2^{(1)} \boldsymbol{\delta}_1^{(1)}) + d_U^{23} (\boldsymbol{\delta}_2^{(1)} \boldsymbol{\delta}_3^{(1)} - \boldsymbol{\delta}_3^{(1)} \boldsymbol{\delta}_2^{(1)}) + d_U^{31} (\boldsymbol{\delta}_3^{(1)} \boldsymbol{\delta}_1^{(1)} - \boldsymbol{\delta}_1^{(1)} \boldsymbol{\delta}_3^{(1)}),$$

$$d_U^{ij} = -2 \, \frac{U_i (\boldsymbol{\delta}_i^{(2)} \cdot e \cdot \boldsymbol{\delta}_j^{(2)}) U_j}{U_i^2 - U_j^2}; \tag{2.3}$$

$$d_V = d_V^{12} (\boldsymbol{\delta}_1^{(2)} \boldsymbol{\delta}_2^{(2)} - \boldsymbol{\delta}_2^{(2)} \boldsymbol{\delta}_1^{(2)}) + d_V^{23} (\boldsymbol{\delta}_2^{(2)} \boldsymbol{\delta}_3^{(2)} - \boldsymbol{\delta}_3^{(2)} \boldsymbol{\delta}_2^{(2)}) + d_V^{31} (\boldsymbol{\delta}_3^{(2)} \boldsymbol{\delta}_1^{(2)} - \boldsymbol{\delta}_1^{(2)} \boldsymbol{\delta}_3^{(2)}),$$

$$d_V^{ij} = \boldsymbol{\delta}_i^{(2)} \cdot d \cdot \boldsymbol{\delta}_j^{(2)} - \frac{(U_i^2 + U_j^2)(\boldsymbol{\delta}_i^{(2)} \cdot e \cdot \boldsymbol{\delta}_j^{(2)})}{U_i^2 - U_j^2}. \tag{2.4}$$

Finally, considering that $\boldsymbol{\delta}_i^{(2)'} = R' \cdot \boldsymbol{\delta}_i^{(1)'}$, $\boldsymbol{\delta}_i^{(1)'} = (g + \varepsilon d_U) \cdot \boldsymbol{\delta}_i^{(1)}$, $\boldsymbol{\delta}_i^{(2)'} = (g + \varepsilon d_V) \cdot \boldsymbol{\delta}_i^{(2)}$, and $\boldsymbol{\delta}_i^{(2)} = R \cdot \boldsymbol{\delta}_i^{(1)}$, we express the orthogonal tensor R' as

$$R' = \Big\{ g + \varepsilon \Big[d - \frac{U_i - U_j}{U_i + U_j} (\boldsymbol{\delta}_i^{(2)} \cdot e \cdot \boldsymbol{\delta}_j^{(2)}) \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_j^{(2)} \Big] \Big\} \cdot R \qquad (2.5)$$

(it is easy to see that the subtracted term in square brackets is a skew-symmetric tensor, similarly to the tensor d). Relations (2.3)–(2.5) imply that the skew-symmetric tensor $d_U = 0$ (the vectors $\boldsymbol{\delta}_i^{(1)}$ do not rotate), the skew-symmetric tensor $d_V = d$ (the vectors $\boldsymbol{\delta}_i^{(2)}$ rotate only due to the deformation-rotation tensor d), and the change of the orthogonal tensor in the polar decomposition of the site gradient is due exclusively to the deformation rotation $R' = (g + \varepsilon d) \cdot R$ only in the case where the eigenvectors of the symmetric tensor of the additional small elastic strain e coincide with the vectors $\boldsymbol{\delta}_i^{(2)}$. In the remaining cases, all these rotations are also affected by the strain tensor.

The exact evolutionary relations for (2.2) and (2.5) are obtained in the same manner as the exact evolutionary relation (1.7) was obtained from the approximate relation (1.6).

If we use the tensor P (4) instead of h, all relations obtained above for elasticity remain valid for the gradients $F_{\rm IN}$ and F_Θ in which $h_{\rm E}$ is replaced by P, $e_{\rm E}$ by P_S, and $d_{\rm E}$ by P_C [$P_S = (P + P^{\rm t})/2$ and $P_C = (P - P^{\rm t})/2$ are the symmetric and skew-symmetric parts P, respectively]. The pure strain tensor present in these relations, their eigenvalues and eigenvectors, and the orthogonal tensor correspond to the inelastic or thermal kinematics, i.e., they have the subscript IN or Θ.

Below, of all kinematic relations, we will need only the expression for the orthogonal tensor of the inelastic and thermal site gradients, which, in the adopted notation is written as

$$R' = \Big\{ g + \varepsilon \Big[P_C - \frac{U_i - U_j}{U_i + U_j} (\boldsymbol{\delta}_i^{(2)} \cdot P_S \cdot \boldsymbol{\delta}_j^{(2)}) \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_j^{(2)} \Big] \Big\} \cdot R \qquad (2.6)$$

(the subscripts IN, Θ, and $*$ are omitted).

3. Inelastic and Thermal Site Gradients without Rotation. As shown in [4], in the case of using decomposition (2), the site gradients $F_{\rm IN}$ and F_Θ should be pure deformations without rotation, i.e., the orthogonal tensors $R_{\rm IN}$ and R_Θ in the polar decompositions of these site gradients should be unit at any time. From this it follows that, in relation (2.6), $R = R' = g$. Then,

$$P_C = \frac{U_i - U_j}{U_i + U_j} (\boldsymbol{\delta}_i^{(2)} \cdot P_S \cdot \boldsymbol{\delta}_j^{(2)}) \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_j^{(2)}. \qquad (3.1)$$

Representing the tensor P_C in the basis $\boldsymbol{\delta}_i^{(2)}$: $P_C = P_C^{ij} \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_j^{(2)}$, where $P_C^{ij} = \boldsymbol{\delta}_i^{(2)} \cdot P_C \cdot \boldsymbol{\delta}_j^{(2)}$, from relation (3.1), we obtain

$$\boldsymbol{\delta}_i^{(2)} \cdot P_C \cdot \boldsymbol{\delta}_j^{(2)} = \frac{U_i - U_j}{U_i + U_j} (\boldsymbol{\delta}_i^{(2)} \cdot P_S \cdot \boldsymbol{\delta}_j^{(2)}).$$

From this it follows that

$$(U_i + U_j)(\boldsymbol{\delta}_i^{(2)} \cdot P_C \cdot \boldsymbol{\delta}_j^{(2)}) = (U_i - U_j)(\boldsymbol{\delta}_i^{(2)} \cdot P_S \cdot \boldsymbol{\delta}_j^{(2)}).$$

Equality of these components implies the equality of the tensors

$$(U_i + U_j)(\boldsymbol{\delta}_i^{(2)} \cdot P_C \cdot \boldsymbol{\delta}_j^{(2)}) \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_j^{(2)} = (U_i - U_j)(\boldsymbol{\delta}_i^{(2)} \cdot P_S \cdot \boldsymbol{\delta}_j^{(2)}) \boldsymbol{\delta}_i^{(2)} \boldsymbol{\delta}_j^{(2)},$$

which is represented as

$$V \cdot P_C + P_C \cdot V = V \cdot P_S - P_S \cdot V. \qquad (3.2)$$

The condition $R = g$ leads to the equality $\boldsymbol{\delta}_i^{(2)} = \boldsymbol{\delta}_i^{(1)}$. This means that $V = U$ and relation (3.2) can be written as

$$U \cdot P_C + P_C \cdot U = U \cdot P_S - P_S \cdot U. \qquad (3.3)$$

Equality (3.3) also follows directly from relation (4). Indeed, if the site gradient is pure strain, then (4) is written as $U = (g + \varepsilon P) \cdot U_*$. Then, the tensor $U^{\rm t} = U_*^{\rm t} \cdot (g + \varepsilon P)^{\rm t}$. Because of the symmetry of the tensors U and U_*, we have $U = U^{\rm t}$ and $U_* = U_*^{\rm t}$. Then, $P \cdot U_* = U_* \cdot P^{\rm t}$. Representing P as the symmetric and skew-symmetric parts, we obtain relation (3.3).

We write the equations for $d_{\rm IN}$ and d_Θ. From expression (4), it follows that $P_{\rm IN} = F_{\rm E*}^{-1} \cdot h_{IN} \cdot F_{\rm E*}$. Determining the symmetric $(P_{\rm IN})_S$ and skew-symmetric $(P_{\rm IN})_C$ parts of the tensor $P_{\rm IN}$ and substituting them into relation (3.3), in which, in this case $U \equiv U_{\rm IN*}$, we obtain the following equation for $d_{\rm IN}$:

$$A \cdot d_{\rm IN} + d_{\rm IN} \cdot A^{\rm t} = C, \qquad A = (F_*^{-})^{\rm t} \cdot U_{\Theta*} \cdot F_{\rm E*}^{-1}, \qquad C = e_{\rm IN} \cdot A^{\rm t} - A \cdot e_{\rm IN}. \tag{3.4}$$

Similarly, substituting the tensor P_Θ from expression (4) and setting $U \equiv U_{\Theta*}$ in (3.3), we obtain the following equation for d_Θ:

$$A \cdot d_\Theta + d_\Theta \cdot A = C, \qquad A = (F_*^{-})^{\rm t} \cdot U_{\Theta*} \cdot F_*^{-1}, \qquad C = e_\Theta \cdot A - A \cdot e_\Theta. \tag{3.5}$$

In Eqs. (3.4) and (3.5), the tensors A and C are known.

Equations (3.4) and (3.5) can be written as $A \cdot X + X \cdot B = C$, where $B = A^{\rm t}$ or $B = A$; $X = d_{\rm IN}$ or $X = d_\Theta$. This equation has a unique solution if the tensors A and $-B$ do not have common eigenvalues (see [6, 7]). Equations (3.4) and (3.5) satisfy this condition. Setting $e_\Theta = \beta\theta g$ in Eq. (3.5), where θ is a small change in the temperature Θ (thermal small deformation obeys linear thermal-expansion law), we obtain $A \cdot d_\Theta + d_\Theta \cdot A = 0$. By virtue of the uniqueness of the solution of this equation, $d_\Theta = 0$. Because the tensor $d_{\rm IN}$ is skew-symmetric, Eq. (3.4) for any basis $\boldsymbol{r}_i$ reduces to the following system of three linear equations with three unknowns:

$$\sum_{\substack{k,l=1\\k<l}}^{3} B_{(ij)}^{kl} d_{kl} = -\sum_{k=1}^{3} C_{(ij)}^{k} e_{kk} - \sum_{\substack{k,l=1\\k<l}}^{3} D_{(ij)}^{kl} e_{kl}, \qquad i,j = 1,2,3, \quad i<j.$$

Here

$$B_{(ij)}^{kl} = A^{ik} g^{lj} + g^{ki} A^{jl} - A^{il} g^{kj} - g^{li} A^{jk},$$

$$C_{(ij)}^{k} = A^{ik} g^{kj} - g^{ki} A^{jk}, \qquad D_{(ij)}^{kl} = A^{ik} g^{lj} - g^{ki} A^{jl} + A^{il} g^{kj} - g^{li} A^{jk},$$

A^{ij} and g^{ij} are the contravariant components of the tensor A from (3.4) and the metric tensor, which are normalized to the basis $\boldsymbol{r}_i$.

We note that passage to the limit reduces Eqs. (3.4) and (3.5) to exact evolutionary equations.

Conclusions. As shown in [4], according to the principle of objectivity, in the representation of the total site gradient F in the form $F = F_{\rm E} \cdot F_{\rm IN} \cdot F_\Theta$, the inelastic and thermal site gradients should be pure deformations without rotations. Using this requirement, the missing relationship was obtained between the small strains $e_{\rm IN}$ with the known constitutive relation and the small rotations $d_{\rm IN}$ and between the small strains e_Θ with the known constitutive relation and the small rotations d_Θ.

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